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The C-matrix and the reality classification of the representations of the homogeneous Lorentz group: I. Irreducible representations of SO(3, 1)

A V Gopala Rao, B S Narahari† and K N Srinivasa Rao‡

Department of Studies in Physics, Manasagangothri, University of Mysore, Mysore 570 006, India

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Abstract. A basis-independent criterion for the classification of irreducible group representations, into potentially-real, pseudo-real and essentially-complex representations, is given for an arbitrary group which may also possess infinite-dimensional representations. These considerations are applied, in particular, to the finite- and infinite-dimensional representations $\mathbf{D}(j_0, c)$ of the orthochronous proper Lorentz group SO(3, 1) and it is shown that the irreps which are neither unitary nor pseudo-unitary are essentially-complex. Further, among the unitary and pseudo-unitary irreps of SO(3, 1), those irreps with a half-odd-integer j_0 are shown to be pseudo-real, while the others with an integer j_0 (including zero) are potentially-real.

1. Introduction

Based on their 'reality', the representations of a group are divided into three categories. A representation $\mathbf{D}$ of a group Γ is of the first kind, or potentially-real, if it is equivalent to a real representation. It is of the second kind, or pseudo-real, if it cannot be brought to a real form, but is still equivalent to the complex-conjugate representation $\mathbf{D}^*$. If $\mathbf{D}$ is not equivalent to $\mathbf{D}^*$, then the representation is of the third kind, or essentially-complex. Although this kind of classification of group representations has been considered in the literature (Wigner 1959, Hamermesh 1964, Lomont 1959) for finite-dimensional representations only, it certainly also provides a useful sorting of infinite-dimensional representations. It is the purpose of this paper to examine how such a classification may be achieved specifically in the case of infinite-dimensional irreducible representations (irreps) of an arbitrary group. The results obtained here not only cover the case of infinite-dimensional irreps, but also yield the corresponding results for the finite-dimensional case in a more exhaustive and invariant form. In fact, we observe that while the existing criterion (see, for example, Wigner (1959), or Hamermesh (1964)) for reality classification refers to the finite-dimensional unitary representations only, the criterion presented here is applicable to unitary as well as non-unitary irreps irrespective of their dimension.

The determination of a matrix $\mathbf{C}$ which intertwines a given representation $\mathbf{D}$ of a group Γ with the complex-conjugate representation $\mathbf{D}^*$ in the sense that

$$\mathbf{C}\mathbf{D}(g) = \mathbf{D}^*(g)\mathbf{C} \tag{1.1}$$

† Present address: Department of Physics, Government First Grade College, Hassan 573 201, India.
‡ Retired Professor of Theoretical Physics.

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for every $\mathbf{D}(g) \in \mathbf{D}$, $g \in \Gamma$, is crucial to the reality classification of group representations. If such a matrix $\mathbf{C}$ exists and is invertible, then representation $\mathbf{D}$ belongs either to the first kind or to the second kind. Otherwise, it belongs to the third kind. Operator $\mathbf{C}$ has been called the time-reversal operator associated with matrix group $\mathbf{D}$ in the literature (Lomont 1959). However, following Wigner (1959), who denotes this operator by $\mathbf{C}$, we shall simply call it the C-matrix (associated with $\mathbf{D}$) to avoid any possible misunderstanding.

In section 1, we show that any non-zero C-matrix intertwining irreps $\mathbf{D}$ and $\mathbf{D}^*$ is necessarily invertible. This observation simplifies the definition of a third-kind irrep. An irrep $\mathbf{D}$ is of the third kind if equation (1.1) can be satisfied *only* by $\mathbf{C} = 0$. Further, we show that a (non-zero) C-matrix is unique except for a constant scalar factor and satisfies *one* and *only one* of the two invariant conditions $\mathbf{C}\mathbf{C}^* = \pm\mathbf{E}$. From these conditions, it follows that the existence of a matrix $\mathbf{C}$, which is expressible as $\mathbf{C} = \mathbf{T}^*\mathbf{T}^{-1}$ in terms of a suitable invertible matrix $\mathbf{T}$, is a necessary and sufficient condition for an irrep to be potentially-real. However, the above criterion is not a useful practical check for potential-reality, especially in the infinite-dimensional case. Instead, we may use the following result proved in this section: the matrix $\mathbf{T} = (\alpha\mathbf{E} + \alpha^*\mathbf{C}^*)$, where α is a complex number, if invertible, is a solution of $\mathbf{C} = \mathbf{T}^*\mathbf{T}^{-1}$ and hence transforms the irrep in question to a real form. We also discuss the problem of the existence of an inverse to matrix $\mathbf{T}$ and point out some special circumstances under which $\mathbf{T}$ possesses an inverse. Our discussion covers, in particular, the case of all the finite-dimensional and infinite-dimensional irreps of SO(3, 1) which is the main object of study in this paper.

In section 3, we show that $\mathbf{C} = \alpha\mathbf{A}^{*-1}\mathbf{G}$, where α is an arbitrary complex number, $\mathbf{G}$ is the bilinear metric and $\mathbf{A}$ is the sesquilinear metric associated with an irrep $\mathbf{D}$ so that the existence of any two of $\mathbf{C}$, $\mathbf{G}$ and $\mathbf{A}$ implies the existence of the other uniquely.

In section 4, we explicitly determine the C-matrices associated with the irreps $\mathbf{D}(j_0, c)$ of SO(3, 1) in the Gelfand–Naimark basis, thus providing a complete reality classification of all the irreps of SO(3, 1).

2. The C-matrix and the criterion for potential-reality of irreps

Since our interest lies primarily with irreps of infinite dimension, we first review some related definitions and concepts. A matrix $\mathbf{T}$ is defined to be non-degenerate if no vector $X \neq 0$ exists such that $\mathbf{T}X = 0$. If $\mathbf{T}$ is a non-degenerate matrix and $X_1 \neq X_2$ are any two distinct vectors, then $\mathbf{T}X_1 \neq \mathbf{T}X_2$ as otherwise $\mathbf{T}X_1 - \mathbf{T}X_2 = \mathbf{T}(X_1 - X_2) = 0$ would require $X_1 = X_2$, which cannot be true in view of the non-degeneracy of $\mathbf{T}$. Therefore, a linear transformation $\mathbf{T}$: $V_1 \to V_2$ from a vector space V_1 to another vector space V_2 is evidently one-to-one whenever $\mathbf{T}$ is non-degenerate. Here, we may also recall that for a linear transformation $\mathbf{T}$: $V_1 \to V_2$ to be invertible, it must provide a one-to-one mapping of V_1 onto V_2.

Next, we note that an infinite-dimensional representation $g \to \mathbf{D}(g)$ of a group Γ acting in a space B is defined to be irreducible (Gelfand *et al* 1963, Gelfand *et al* 1966) if B has no proper subspaces which are invariant under all the operators $\mathbf{D}(g)$ and any bounded linear operator $\mathbf{K}$ which commutes with all the operators $\mathbf{D}(g)$ is a multiple of the unit operator $\mathbf{E}$. The absence of proper invariant subspaces in B relative to $\mathbf{D}(g)$ is referred to as the subspace-irreducibility of $\mathbf{D}(g)$. The second property of $\mathbf{D}(g)$ relating to the commuting operator $\mathbf{K}$ mentioned above is referred to as the operator-irreducibility of $\mathbf{D}(g)$. For finite-dimensional representations, as is well known, operator irreducibility is equivalent to subspace irreducibility, which, however, is not true of infinite-dimensional irreps in general (see Gelfand *et al* (1966)).

A matrix $\mathbf{T}$ is said to intertwine two representations $\mathbf{D}_1$ and $\mathbf{D}_2$ of a group Γ if $\mathbf{T}\mathbf{D}_1(g) = \mathbf{D}_2(g)\mathbf{T}$ for all $g \in \Gamma$. Further, when the representations $\mathbf{D}_1$ and $\mathbf{D}_2$ are subspace irreducible, it follows, as a consequence of the generalized Schur lemma (Coleman 1968, Mackey 1978), that any matrix $\mathbf{T} \neq 0$ that intertwines the (subspace irreducible) representations $\mathbf{D}_1$ and $\mathbf{D}_2$ of Γ is necessarily invertible, so that $\mathbf{D}_1$ and $\mathbf{D}_2$ are actually equivalent.

Recall that, by definition, a C-matrix intertwines a representation $\mathbf{D}$ of a group Γ with the complex conjugate representation $\mathbf{D}^*$ (see equation (1.1)). When $\mathbf{D}$ and $\mathbf{D}^*$ are irreducible†, every matrix $\mathbf{C} \neq 0$ that intertwines $\mathbf{D}$ with $\mathbf{D}^*$ is necessarily invertible by the Schur lemma. Thus, if $\mathbf{D}$ is an irrep of the third kind, then $\mathbf{C} = 0$ is the *only* matrix that can 'intertwine' $\mathbf{D}$ with $\mathbf{D}^*$.

Suppose a non-zero matrix $\mathbf{C}$ intertwines the irreps $\mathbf{D}$ and $\mathbf{D}^*$. Then, $\mathbf{D}$ can be of the first or second kind only. Moreover, such a matrix $\mathbf{C}$ is necessarily invertible and unique to within a scalar factor. To see this, observe that if $\mathbf{C}_1$ and $\mathbf{C}_2$ are any two non-zero C-matrices which intertwine the equivalent irreps $\mathbf{D}$ and $\mathbf{D}^*$, then we must have

$$\mathbf{C}_1\mathbf{D}(g)\mathbf{C}_1^{-1} = \mathbf{D}^*(g) = \mathbf{C}_2\mathbf{D}(g)\mathbf{C}_2^{-1}$$

so that

$$\mathbf{D}(g)\mathbf{C}_1^{-1}\mathbf{C}_2 = \mathbf{C}_1^{-1}\mathbf{C}_2\mathbf{D}(g).$$

Since $\mathbf{D}$ is operator irreducible, it follows from the Schur lemma that $\mathbf{C}_1^{-1}\mathbf{C}_2 = \alpha\mathbf{E}$, or $\mathbf{C}_2 = \alpha\mathbf{C}_1$, where α is a scalar, thus proving our assertion. In addition to the C-matrix $\mathbf{C}$, we introduce later in this paper three more matrices $\mathbf{T}$, $\mathbf{G}$, and $\mathbf{A}$, and we emphasize that we have assumed that the matrices $\mathbf{C}$, $\mathbf{T}$, $\mathbf{G}$ and $\mathbf{A}$, together with their inverses, are associative among themselves and with the matrices $\mathbf{D}(g)$ of the irrep $\mathbf{D}$. Substituting for $\mathbf{D}^*(g)$ from equation (1.1) in the complex conjugate of equation (1.1), we get

$$\mathbf{D}(g)\mathbf{C}^* = \mathbf{C}^*\mathbf{D}^*(g) = \mathbf{C}^*\mathbf{C}\mathbf{D}(g)\mathbf{C}^{-1}$$

which gives

$$\mathbf{D}(g)\mathbf{C}^*\mathbf{C} = \mathbf{C}^*\mathbf{C}\mathbf{D}(g). \tag{2.1}$$

Since $\mathbf{D}$ is irreducible, this implies that

$$\mathbf{C}^*\mathbf{C} = \beta\mathbf{E} \tag{2.2}$$

where β is a scalar. Taking the complex conjugate of the above equation, we obtain $\mathbf{C}\mathbf{C}^* = \beta^*\mathbf{E}$, so that $\mathbf{C}^* = \beta^*\mathbf{C}^{-1}$. Using this in equation (2.2), we obtain

$$\beta = \beta^* \tag{2.3}$$

and, hence,

$$\mathbf{C}\mathbf{C}^* = \mathbf{C}^*\mathbf{C} = \beta\mathbf{E} = \pm|\beta|\mathbf{E}. \tag{2.4}$$

† Here we wish to point out that throughout this paper, the terms 'irreducible representation' and 'irrep', unless otherwise stated, refer always to a representation which is both subspace and operator irreducible.

Since $\mathbf{C}$ is determined only up to a scalar factor, we may absorb $\sqrt{|\beta|}$ into $\mathbf{C}$ and redefine it without loss of generality to satisfy the condition

$$\mathbf{C}\mathbf{C}^* = \mathbf{C}^*\mathbf{C} = \pm\mathbf{E}. \tag{2.5}$$

Such a C-matrix is evidently determined only up to an arbitrary phase factor $\exp(i\varphi)$, where φ is a real number. Note that a C-matrix which satisfies $\mathbf{C}\mathbf{C}^* = \mathbf{E}$ cannot be made to satisfy $\mathbf{C}\mathbf{C}^* = -\mathbf{E}$ and *vice versa*, by multiplying it by a phase factor which is the only indeterminacy left in $\mathbf{C}$. Hence, it follows that the C-matrix can always be chosen so as to satisfy *one* (and *only one*) of the two conditions in equation (2.5). Further, the conditions given by equation (2.5) are easily seen to be invariant under a change of basis in the carrier space of the irrep $\mathbf{D}$. Let $\mathbf{D}(g) \to \mathbf{D}'(g) = \mathbf{S}^{-1}\mathbf{D}(g)\mathbf{S}$ under a change of basis. Writing $\mathbf{D}(g) = \mathbf{S}\mathbf{D}'(g)\mathbf{S}^{-1}$ in equation (1.1) and rearranging, we obtain

$$\mathbf{C}'\mathbf{D}'(g) = \mathbf{D}'^*(g)\mathbf{C}' \tag{2.6}$$

where we have defined

$$\mathbf{C}' \equiv \mathbf{S}^{*-1}\mathbf{C}\mathbf{S} \tag{2.7}$$

and such a $\mathbf{C}'$ satisfies $\mathbf{C}'\mathbf{C}'^* = \pm\mathbf{E}$ again, thus proving our assertion. Therefore, the totality of irreps of the first and second kind may be partitioned into two mutually-exclusive sets in an invariant manner depending on whether the associated C-matrices satisfy $\mathbf{C}\mathbf{C}^* = \mathbf{E}$ or $\mathbf{C}\mathbf{C}^* = -\mathbf{E}$. We may also note that equation (2.7) is, incidentally, the rule by which the C-matrix transforms under a change of basis.

Consider an irrep $\mathbf{D}$ of the first kind. Then, by definition, a similarity transformation $\mathbf{T}$ exists such that $\mathbf{T}^{-1}\mathbf{D}(g)\mathbf{T}$ is real. Then,

$$\mathbf{T}^{-1}\mathbf{D}(g)\mathbf{T} = (\mathbf{T}^{-1}\mathbf{D}(g)\mathbf{T})^* = \mathbf{T}^{*-1}\mathbf{D}^*(g)\mathbf{T}^* \tag{2.8a}$$

so that

$$\mathbf{T}^*\mathbf{T}^{-1}\mathbf{D}(g) = \mathbf{D}^*(g)\mathbf{T}^*\mathbf{T}^{-1}. \tag{2.8b}$$

Comparing this with equation (1.1) and taking into account the 'uniqueness' of $\mathbf{C}$, we obtain

$$\mathbf{C} = \mathbf{T}^*\mathbf{T}^{-1} \tag{2.9}$$

to within an arbitrary phase factor. This $\mathbf{C}$ evidently satisfies $\mathbf{C}\mathbf{C}^* = \mathbf{E}$ only and, hence, it follows that the existence of a C-matrix satisfying $\mathbf{C}\mathbf{C}^* = \mathbf{E}$ is a similarity-invariant *necessary* condition for an irrep to be potentially-real.

On the other hand, an irrep $\mathbf{D}$ whose C-matrix satisfies $\mathbf{C}\mathbf{C}^* = -\mathbf{E}$ cannot possess a $\mathbf{T}$ which sends it to a real form since otherwise such a $\mathbf{T}$ would generate a $\mathbf{C}$ through equation (2.9) which satisfies $\mathbf{C}\mathbf{C}^* = \mathbf{E}$. Therefore, the existence of a C-matrix satisfying the condition $\mathbf{C}\mathbf{C}^* = -\mathbf{E}$ is a similarity-invariant *sufficient* condition for an irrep to be pseudo-real. These results also show that the existence of a C-matrix expressible as $\mathbf{C} = \mathbf{T}^*\mathbf{T}^{-1}$ is both necessary and sufficient for an irrep to be potentially-real. It must be observed here that the above results are true of both finite- and infinite-dimensional irreps.

We now examine the conditions under which a given C-matrix may be expressed as $\mathbf{C} = \mathbf{T}^*\mathbf{T}^{-1}$. Such a C-matrix evidently satisfies $\mathbf{CC}^* = \mathbf{E}$. Unlike the case of finite-dimensional matrices, we do not know how to express the inverse $\mathbf{T}^{-1}$ of an infinite-dimensional matrix explicitly as a function of the elements of $\mathbf{T}$. Therefore, we rewrite equation (2.9) as

$$\mathbf{CT} = \mathbf{T}^* \tag{2.10}$$

where we must remember that $\mathbf{C}$ satisfies $\mathbf{CC}^* = \mathbf{E}$. However, we must be careful because, while equation (2.9) admits only invertible solutions $\mathbf{T}$, related equation (2.10) also admits degenerate solutions (including $\mathbf{T} = 0$). It is easily checked that the following one-parameter family of matrices

$$\mathbf{T} = \alpha + \alpha^*\mathbf{C}^* \tag{2.11}$$

where α is an arbitrary complex number, satisfies equation (2.10). Further, such a $\mathbf{T}$-matrix would be non-degenerate whenever $(-\alpha/\alpha^*)$ is not an eigenvalue of the C-matrix. To see this, observe that

$$\mathbf{T}X = \alpha[\mathbf{E} + (\alpha^*/\alpha)\mathbf{C}^*]X = 0$$

would possess non-trivial solutions X if the equation

$$\mathbf{C}^*X = -(\alpha/\alpha^*)X$$

admits, in turn, solutions $X \neq 0$. In other words, a $\mathbf{T}$ given by equation (2.11) would be non-degenerate if, and only if, $(-\alpha/\alpha^*)$ is *not* an eigenvalue of the matrix $\mathbf{C}$.

If $\mathbf{C}$ is a finite-dimensional matrix, then, since there are only a finite number of eigenvalues for it, we can certainly find a complex number α such that $(-\alpha/\alpha^*)$ is not an eigenvalue of $\mathbf{C}$. Therefore, in the finite-dimensional case, equation (2.11), with an appropriate choice of α, gives a non-degenerate, and hence invertible, solution $\mathbf{T}$ of equation (2.10). We thus arrive at the following theorem.

Theorem 1a. The existence of a C-matrix satisfying $\mathbf{CC}^* = \mathbf{E}$ is a similarity-invariant necessary and sufficient condition for a finite-dimensional irrep to be potentially-real.

In view of the fact that $\mathbf{C}$ must satisfy only one of the invariant conditions $\mathbf{CC}^* = \mathbf{E}$ or $\mathbf{CC}^* = -\mathbf{E}$, the above theorem implies, in turn, the following.

Theorem 1b. The existence of a C-matrix satisfying $\mathbf{CC}^* = -\mathbf{E}$ is a similarity-invariant necessary and sufficient condition for a finite-dimensional irrep to be pseudo-real.

In the infinite-dimensional case, on the other hand, to ensure that $(-\alpha/\alpha^*)$ is not an eigenvalue of $\mathbf{C}$, we must have a knowledge of the eigenvalue spectrum of $\mathbf{C}$, which is not easy to obtain since all that we know about $\mathbf{C}$ is that it is a discrete infinite-dimensional matrix satisfying $\mathbf{CC}^* = \mathbf{E}$. Even if we succeed in identifying a number $(-\alpha/\alpha^*)$ which is not an eigenvalue of $\mathbf{C}$, it may not solve the problem every time since the $\mathbf{T}$ obtained using equation (2.11) could still be non-invertible, although non-degenerate. (This is so because a non-degenerate matrix $\mathbf{T}$ affects only a one-to-one mapping of the vectors of the carrier space B of the irrep $\mathbf{D}$. In order that $\mathbf{T}$ be invertible, it must be a one-to-one mapping of B *onto* B.) However, the problem readily admits a solution in one special case, which covers the group under study, namely SO(3, 1). Suppose $\mathbf{C}$ is real in some basis of B. Then, the (invariant) condition $\mathbf{CC}^* = \mathbf{E}$ becomes $\mathbf{C}^2 = \mathbf{E}$ in that basis. This would imply that every eigenvalue of $\mathbf{C}$ is either +1 or -1 and, hence, with any $\alpha \neq \pm\alpha^*$, we obtain a non-degenerate matrix $\mathbf{T}$ from equation (2.11). Moreover, every such $\mathbf{T} = \alpha + \alpha^*\mathbf{C}$, where $\alpha \neq \pm\alpha^*$, is invertible. Assuming $\mathbf{T}^{-1} = \beta + \gamma\mathbf{C}$, we find that $\mathbf{T}^{-1}\mathbf{T} = \mathbf{T}\mathbf{T}^{-1} = \mathbf{E}$, if $\beta = \alpha/(\alpha^2 - \alpha^{*2})$ and $\gamma = -\alpha^*/(\alpha^2 - \alpha^{*2})$. We choose $\alpha = (1-\mathrm{i})/2$ and cast $\mathbf{T}$ and $\mathbf{T}^{-1}$ in the convenient forms

$$\mathbf{T} = [(1-\mathrm{i})/2](\mathbf{E} + \mathrm{i}\mathbf{C}) \qquad \mathbf{T}^{-1} = [(1+\mathrm{i})/2](\mathbf{E} - \mathrm{i}\mathbf{C}). \tag{2.12}$$

Therefore, if the C-matrix is real in some basis, then in that basis, the conditions

$$\mathbf{C}^2 = \mathbf{E} \Leftrightarrow \text{Potential-reality} \qquad \text{and} \qquad \mathbf{C}^2 = -\mathbf{E} \Leftrightarrow \text{Pseudo-reality}.$$

Although generally not useful, the above conditions serve the purpose of this paper very well, because, as will be shown later in section 4, the invertible C-matrices associated with the first- and second-kind irreps of SO(3, 1) are all real in the Gelfand–Naimark basis.

3. The nature of the C-matrix associated with metric-preserving irreps

When an irreducible matrix group $\mathbf{D}$ possessing a C-matrix $\mathbf{C}$ also possesses bilinear and sesquilinear metrics $\mathbf{G}$ and $\mathbf{A}$, the three matrices $\mathbf{C}$, $\mathbf{G}$ and $\mathbf{A}$ are interrelated as a consequence of the irreducibility of $\mathbf{D}$. We must, however, note that all three matrices $\mathbf{C}$, $\mathbf{G}$ and $\mathbf{A}$ need not exist simultaneously for an irreducible matrix group. In fact, there are irreducible matrix groups for which none of these three matrices exist while there are others which possess only one of them. For example, while the self-representation of the matrix group GL(2, C) does not possess any of these matrices, the self-representations of the groups GL(2, R), SL(2, C) and SU(n), when $n > 2$, possess only one of these matrices $\mathbf{C}$, $\mathbf{G}$ and $\mathbf{A}$, in that order. However, we prove, in the sequel, that the simultaneous existence of any two of $\mathbf{C}$, $\mathbf{G}$ and $\mathbf{A}$ for an irrep implies the existence of the third uniquely, except for an arbitrary scalar factor.

First, we recall some well known properties of the bilinear and sesquilinear metrics associated with irreducible matrix groups. A matrix group (or representation) $\mathbf{D}$, with elements $\mathbf{D}(g)$, is said to preserve a bilinear metric if there exists a non-degenerate matrix $\mathbf{G}$ such that (see, for example, Gilmore (1974))

$$\tilde{\mathbf{D}}(g)\mathbf{G}\mathbf{D}(g) = \mathbf{G} \qquad \forall \mathbf{D}(g) \in \mathbf{D} \tag{3.1}$$

where the tilde ˜ denotes matrix transposition. Such a bilinear metric $\mathbf{G}$ *associated with* an irreducible matrix group $\mathbf{D}$ is determined uniquely up to a scalar factor and can either be symmetric or anti-symmetric only (Freudenthal and de Vries 1969), i.e. $\tilde{\mathbf{G}} = \pm\mathbf{G}$. By definition, a matrix group (or representation) $\mathbf{D}$ which preserves a symmetric bilinear metric is called orthogonal, whereas one which preserves an antisymmetric bilinear metric is called symplectic. Similarly, a matrix group (or representation) $\mathbf{D}$, with elements $\mathbf{D}(g)$ is said to preserve a sesquilinear metric if there exists a non-degenerate matrix $\mathbf{A}$ such that

$$\mathbf{D}^{\dagger}(g)\mathbf{A}\mathbf{D}(g) = \mathbf{A} \qquad \forall \mathbf{D}(g) \in \mathbf{D} \tag{3.2}$$

where $\mathbf{D}^\dagger \equiv \tilde{\mathbf{D}}^*$ is the hermitian conjugate of $\mathbf{D}$. The sesquilinear metric $\mathbf{A}$ associated with an irreducible matrix group (or representation) $\mathbf{D}$ is also uniquely determined, except for a scalar factor, and can always be chosen to be hermitian (Freudenthal and de Vries 1969). A matrix group (or representation) which preserves a sesquilinear metric $\mathbf{A}$ is said to be unitary if $\mathbf{A}$ is positive-definite and pseudo-unitary if $\mathbf{A}$ is indefinite.

We now prove that if any two of the matrices $\mathbf{C}$, $\mathbf{G}$ and $\mathbf{A}$ exist for an irrep $\mathbf{D}$, then the third certainly exists as a simple function of the other two.

(i) Let an irrep $\mathbf{D}$ admit $\mathbf{G}$ and $\mathbf{A}$. Then, from equations (3.1) and (3.2), we get

$$\mathbf{D}^*(g)\mathbf{A}^{*-1}\mathbf{G} = \mathbf{D}^*(g)[\mathbf{D}^{*-1}(g)\mathbf{A}^{*-1}\tilde{\mathbf{D}}^{-1}(g)][\tilde{\mathbf{D}}(g)\mathbf{G}\mathbf{D}(g)] = \mathbf{A}^{*-1}\mathbf{G}\mathbf{D}(g). \tag{3.3}$$

Thus,

$$\mathbf{C} = \alpha\mathbf{A}^{*-1}\mathbf{G} \tag{3.4}$$

is the C-matrix associated with $\mathbf{D}$ where α is any scalar factor.

(ii) On the other hand, if an irrep $\mathbf{D}$ admits $\mathbf{A}$ and $\mathbf{C}$, then we have from equation (1.1) $\mathbf{C} = \mathbf{D}^{*-1}(g)\mathbf{C}\mathbf{D}(g)$, which, when multiplied on the left-hand side by $\mathbf{A}^*$ from equation (3.2), yields

$$\mathbf{A}^*\mathbf{C} = [\tilde{\mathbf{D}}(g)\mathbf{A}^*\mathbf{D}^*(g)][\mathbf{D}^{*-1}(g)\mathbf{C}\mathbf{D}(g)] = \tilde{\mathbf{D}}(g)\mathbf{A}^*\mathbf{C}\mathbf{D}(g). \tag{3.5}$$

Comparing this with equation (3.1), we see that

$$\mathbf{G} = \beta\mathbf{A}^*\mathbf{C} \tag{3.6}$$

is the bilinear metric preserved by $\mathbf{D}$, where β is a scalar. Since $\tilde{\mathbf{G}} = \pm\mathbf{G}$, in general, this implies that the matrix $\mathbf{A}^*\mathbf{C}$ is also symmetric or skew-symmetric accordingly.

(iii) Finally, when an irrep $\mathbf{D}$ possesses $\mathbf{C}$ and $\mathbf{G}$, we have, from equation (1.1), $\mathbf{C} = \mathbf{D}^{*-1}(g)\mathbf{C}\mathbf{D}(g)$, which, together with equation (3.1), implies that

$$\mathbf{G}^*\mathbf{C}^{*-1} = [\tilde{\mathbf{D}}^*(g)\mathbf{G}^*\mathbf{D}^*(g)][\mathbf{D}^{*-1}(g)\mathbf{C}^{*-1}\mathbf{D}(g)] = \mathbf{D}^\dagger(g)\mathbf{G}^*\mathbf{C}^{*-1}\mathbf{D}(g).$$

Thus,

$$\mathbf{A} = \gamma\mathbf{G}^*\mathbf{C}^{*-1} \tag{3.7}$$

is the sesquilinear metric preserved by $\mathbf{D}$, where γ is a scalar. With a suitable choice of γ, this $\mathbf{A}$ can be made hermitian.

We may note that the defining relations (3.1), (3.2) and (1.1) of $\mathbf{G}$, $\mathbf{A}$ and $\mathbf{C}$, respectively, may be expressed in a more useful form in terms of the infinitesimal generators $\mathbf{I}_k$, $k = 1, 2, \ldots, n$ of the matrix representation $\mathbf{D}$ as follows

$$\mathbf{G}\mathbf{I}_k = -\tilde{\mathbf{I}}_k\mathbf{G} \qquad \mathbf{A}\mathbf{I}_k = -\mathbf{I}_k^\dagger\mathbf{A} \qquad \mathbf{C}\mathbf{I}_k = \mathbf{I}_k^*\mathbf{C} \qquad k = 1, 2, \ldots, n. \tag{3.8}$$

It is not difficult to check that these equations also lead to the same relations (2.5), (3.4), (3.6) and (3.7), as they must for reasons of consistency. In an actual determination of $\mathbf{G}$, $\mathbf{C}$ or $\mathbf{A}$ associated with a given matrix group (or representation), one must invariably use equations (3.8) only. Although there are n equations in (3.8), generally, not all would be independent because of the commutation relations satisfied by the $\mathbf{l}_k$, thus simplifying the problem a little.

We perform the reality classification of the irreps of SO(3, 1) in the next section, making use of the interrelationship of $\mathbf{G}$, $\mathbf{C}$ and $\mathbf{A}$, discussed above. Immediately, we note here another useful application of equation (3.4). In the case of unitary irreps for which $\mathbf{A} = \mathbf{E}$ in appropriate bases, the C-matrix may also be chosen to be unitary. To see this, we note that when $\mathbf{A} = \mathbf{E}$, equation (3.4) implies that $\mathbf{C} = \alpha\mathbf{G}$ (where α is chosen so that $\mathbf{CC}^* = \pm\mathbf{E}$). However, in general, $\tilde{\mathbf{G}} = \pm\mathbf{G}$ and hence it follows that $\tilde{\mathbf{C}} = \pm\mathbf{C}$ accordingly. As such, $\mathbf{CC}^\dagger = \mathbf{C}\tilde{\mathbf{C}}^* = \pm\mathbf{CC}^* = \pm\mathbf{E}$. However, every diagonal element of a matrix of type $\mathbf{MM}^\dagger$ is non-negative and, hence, $\mathbf{CC}^\dagger$ can never equal $-\mathbf{E}$. Thus, $\mathbf{CC}^\dagger = \mathbf{E}$, i.e. $\mathbf{C}$ is unitary. Therefore, for a unitary irrep, a unitary symmetric $\mathbf{C} \Leftrightarrow \mathbf{CC}^* = \mathbf{E}$ and a unitary skew-symmetric $\mathbf{C} \Leftrightarrow \mathbf{CC}^* = -\mathbf{E}$. Now, from theorems (1*a*) and (1*b*), we obtain the following well known result that a finite-dimensional unitary irrep is potentially-real or pseudo-real according to

$$\mathbf{CC}^\dagger = \mathbf{E} \qquad \tilde{\mathbf{C}} = \mathbf{C} \qquad \text{or} \qquad \mathbf{CC}^\dagger = \mathbf{E} \qquad \tilde{\mathbf{C}} = -\mathbf{C}. \tag{3.9}$$

In passing, we note that the above criterion (3.9) is not applicable to finite-dimensional irreps which are not equivalent to unitary irreps (such as the finite-dimensional irreps of SO(3, 1)). Second, even in the case of finite-dimensional unitary irreps, the above criterion (3.9) is not useful in an arbitrary basis, since the C-matrix would not be unitary in all bases in view of its transformation law (2.7). Therefore, it is certainly advantageous to also use the basis-independent criteria $\mathbf{CC}^* = \pm\mathbf{E}$ in the reality classification of finite-dimensional irreps.

4. The reality-classification of the irreps of SO(3, 1)

Not all the irreps $\mathbf{D}(j_0, c)$ of SO(3, 1) preserve a non-degenerate sesquilinear metric. To quote the results from the book of Gelfand *et al* (1963) in this context, we conveniently break up the set of all the $\mathbf{D}(j_0, c)$ irreps of SO(3, 1) as the union of the disjoint subsets U_1, U_2, U_3, PU_1, PU_2, NU_1 and NU_2 which are defined in table 1. Then the results of Gelfand *et al* (1963) are:

(i) all the irreps contained in $U_1 \cup U_2 \cup U_3$ are unitary;

(ii) all the irreps contained in $PU_1 \cup PU_2$ are pseudo-unitary; and

(iii) the rest of the irreps which are all contained in $NU_1 \cup NU_2$ are non-unitary (in the sense that they do not preserve any sesquilinear metric $\mathbf{A}$).

Further, we may note that $U_1 \cup U_2$ is more familiarly known as the principal series and U_3 as the complementary series (of unitary representations) of SO(3.1).

In an earlier paper (Rao *et al* 1983), we have shown that *every* irrep $\mathbf{D}(j_0, c)$ of SO(3, 1) preserves a non-degenerate bilinear metric $\mathbf{G}$ (in the sense of the first of equations (3.8)), which can be chosen to be real in the Gelfand–Naimark basis. Therefore, it is obvious that every irrep belonging to the union of U_1, U_2, U_3, PU_1 and PU_2 possesses both $\mathbf{A}$ and $\mathbf{G}$ and, hence, a $\mathbf{C}$ also, through equation (3.4). The remaining irreps of SO(3, 1) contained in $NU_1 \cup NU_2$ possess only a $\mathbf{G}$, and as these irreps do not preserve any $\mathbf{A}$, it follows from the interrelationships (3.4), (3.6) and (3.7) that these irreps must be essentially complex (called type III in table 1). Choosing the arbitrary complex scalar α in equation (3.4) to be unity and using the special properties

$$\mathbf{A} = \mathbf{A}^* = \tilde{\mathbf{A}} = \mathbf{A}^{-1} \qquad \mathbf{G} = \mathbf{G}^* = \pm\tilde{\mathbf{G}} = \pm\mathbf{G}^{-1} \qquad \mathbf{AG} = \mathbf{GA} \tag{4.1}$$

Table 1. Reality classification of the irreps of SO(3, 1) where: the matrix elements of the bilinear metric **G** are $G(j', m'; j, m) = (-1)^{m-j_0}\delta_{j'j}\delta_{m',-m}$; the matrix elements of **G**, **A** and **C** are given in the Gelfand–Naimark basis; n is any natural number, $1, 2, 3\ldots$; $k = j$ for $j \leqslant [c]$ and $k = c$ for $j > [c]$, where $[c]$ is the integral part of c; r' is any real number $\neq j_0 + n$; and c' is any complex number with $\mathrm{Re}(c) \neq 0$ and $\mathrm{Im}(c) \neq 0$.

	Parameter Range			Sesquilinear metric	C-matrix	
Name of irrep	j_0	c	Dimension of irrep	$A(j', m'; j, m)$	$C(j', m'; j, m)$	Reality type
U_1	0 or n	pure imaginary	infinite	$\delta_{j',j}\delta_{m',m}$	$(-1)^m\delta_{j',j}\delta_{m',-m}$	I
U_2	$(2n-1)/2$	pure imaginary	infinite	$\delta_{j',j}\delta_{m',m}$	$\mathrm{i}(-1)^m\delta_{j',j}\delta_{m',-m}$	II
U_3	0	$0 < c \leqslant 1$	infinite	$\delta_{j',j}\delta_{m',m}$	$(-1)^m\delta_{j',j}\delta_{m',-m}$	I
PU_1	0	$n+1$	c^2	$(-1)^j\delta_{j',j}\delta_{m',m}$	$(-1)^{m+j}\delta_{j',j}\delta_{m',-m}$	I
PU_2	0	any non-integer in $1 < c < \infty$	infinite	$(-1)^k\delta_{j',j}\delta_{m',m}$	$(-1)^{k+m}\delta_{j',j}\delta_{m',-m}$	I
NU_1	n or $(2n-1)/2$	$c = j_0 + n$	n	does not exist	does not exist	III
		$c = r'$	infinite			
NU_2	0, n or $(2n-1)/2$	$c = r'$ or $c = c'$	infinite	does not exist	does not exist	III

of **A** and **G** valid in the GN-basis (see table 1), we obtain the following special properties of **C** in the GN-basis for irreps not belonging to $NU_1 \cup NU_2$:

$$\mathbf{C} = \mathbf{AG} = \mathbf{GA} \tag{4.2}$$

$$\mathbf{C} = \mathbf{C}^* = \pm\tilde{\mathbf{C}} = \pm\mathbf{C}^{-1} \qquad \mathbf{CC}^* = \mathbf{C}^*\mathbf{C} = \mathbf{C}^2 = \pm\mathbf{E} \tag{4.3}$$

$$\mathbf{CC}^\dagger = \mathbf{C}^\dagger\mathbf{C} = \mathbf{E}. \tag{4.4}$$

Note that in equations (4.1)–(4.4), the plus sign refers to the irreps belonging to U_1, U_3, PU_1 and PU_2 for which j_0 is an integer or zero, and the minus sign refers to the U_2-irreps for which j_0 is half-odd-integral. The fact that it has been possible to choose the C-matrices to be unitary reflects yet another advantage of using the GN-basis. Now, observing that the C-matrices are all real and satisfy $\mathbf{C}^2 = \pm\mathbf{E}$, we conclude, in the light of the remarks made at the end of section 2, that while the irreps belonging to the sub-classes U_1, U_3, PU_1 and PU_2 are potentially real (called type-I in table 1), those belonging to U_2 are pseudo-real (called type-II in table 1). The rest of the irreps of SO(3, 1), which are all contained in $NU_1 \cup NU_2$, are essentially-complex (type-III).

We also note in passing that the C-matrices given in table 1 (in the GN-basis) are unitary (in fact, real orthogonal) and using them in equation (2.12) yields the matrices **T** *which transform the type-I irreps (given in the GN-basis) to their corresponding real forms.* Using these matrices **T**, it is also easy to see that the U_1 and U_3 irreps are real-orthogonal whereas the PU_1 and PU_2 irreps are real pseudo-orthogonal. (For the signatures of the finite-dimensional real pseudo-orthogonal PU_{1-} irreps, see Rao *et al* (1983).)

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